Anomalous scaling in the anisotropic sectors of the Kraichnan model of passive scalar advection

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Kraichnan's model of passive scalar advection in which the driving (Gaussian) velocity field has fast temporal decorrelation is studied as a case model for understanding the anomalous scaling behavior in the anisotropic sectors of turbulent fields. We show here that the solutions of the Kraichnan equation for the n-order correlation functions foliate into sectors that are classified by the irreducible representations of the SO(d) symmetry group. We find a discrete spectrum of universal anomalous exponents, with a different exponent characterizing the scaling behavior in every sector. Generically the correlation functions and structure functions appear as sums over all these contributions, with nonuniversal amplitudes that are determined by the anisotropic boundary conditions. The isotropic sector is always characterized by the smallest exponent, and therefore for sufficiently small scales local isotropy is always restored. The calculation of the anomalous exponents is done in two complementary ways. In the first they are obtained from the analysis of the correlation functions of gradient fields. The theory of these functions involves the control of logarithmic divergences that translate into anomalous scaling with the ratio of the inner and the outer scales appearing in the final result. In the second method we compute the exponents from the zero modes of the Kraichnan equation for the correlation functions of the scalar field itself. In this case the renormalization scale is the outer scale. The two approaches lead to the same scaling exponents for the same statistical objects, illuminating the relative role of the outer and inner scales as renormalization scales. In addition we derive exact fusion rules, which govern the small scale asymptotics of the correlation functions in all the sectors of the symmetry group and in all dimensions.

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I. INTRODUCTION

The aim of this paper is twofold. On the one hand, we are interested in the effects of anisotropy on the universal aspects of scaling behavior in turbulent systems. To this aim we present below a theory of the anomalous scaling of the Kraichnan model of turbulent advection [1] in anisotropic sectors that are classified by the irreducible representations of the SO(d) symmetry group. On the other hand, we are interested in clarifying the relationship between ultraviolet and infrared anomalies in turbulent systems. Again, it turns out that the Kraichnan model is an excellent case model in which this relationship can be exposed with complete clarity. As is well known by now, the Kraichnan model describes the advection of a passive scalar by a velocity field that is random, Gaussian, and δ correlated in time. The correlation functions of the field scale in their spatial dependence, and the main question is what are the scaling (or homogeneity) exponents of the statistical objects of the scalar field that are induced by the given value of the scaling exponent of the advecting velocity field.

The two issues discussed in this paper have an importance that transcends the particular example that we treat in detail in this paper. The first is the role of anisotropy in the observed scaling properties in turbulence. We have shown recently that in the presence of anisotropic effects (which are ubiquitous in realistic turbulent systems) one needs to carefully disentangle the various universal scaling contributions. Even at the largest available Reynolds numbers the observed scaling behavior is not simple, being composed of several contributions with different scaling exponents. The statistical objects like structure functions and correlations functions are characterized by one scaling (or homogeneity) exponent only in the idealized case of full isotropy, or infinite Reynolds numbers when the scaling regime is of infinite extent. Anisotropy results in mixing in various contributions to the statistical objects, each of which is characterized by one universal exponent, but the total is a sum of such contributions that appears not to ''scale'' in standard log-log plots. By realizing that the correlation functions have natural projections on the irreducible representations of the SO(3) symmetry group we could offer methods of data analysis that allow one to measure the universal scaling exponents in each sector separately. In this paper we show that this foliation is an exact property of the statistical objects that arise in the context of the Kraichnan model.

The second issue that transcends the particular example of the Kraichnan model is the identification of the renormalization scales that are associated with anomalous exponents. As has been already explained before, the renormalization scale that appears in the correlation functions of the passively advected scalar field is the outer scale of turbulence L. On the other hand, the theory of correlations of gradients of the field expose in the inner (or dissipative) scale an additional renormalization scale. Having below a theory of anomalous scaling in various sectors of the symmetry group allows us to explain clearly the relationship between the two renormalization scales and the anomalous exponents that are implied by their existence. Since we expect that Kolmogorov-type theories, which assume that no renormalization scale appears in the theory, are generally invalidated by the appearance of both the outer and the inner scales as renormalization scales,

the clarification of the relation between the two is important also for other cases of turbulent statistics.

The central quantitative result of this paper is the expression for the scaling exponent $\zeta_n^{(l)}$, which is associated with the scaling behavior of the n-order correlation function (or structure function) of the scalar field in the lth sector of the symmetry group. In other words, this is the scaling exponent of the projection of the correlation function on the lth irreducible representation of the SO(d) symmetry group, with n and l taking on even values only, $n=0,2,...$ and $l=0,2,...$,

$$\zeta_n^{(l)} = n - \epsilon\left[\frac{n(n+d)}{2(d+2)} - \frac{(d+1)l(l+d-2)}{2(d+2)(d-1)}\right] + O(\epsilon^2). \tag{1.1}$$

The result is valid for any even $l \leqslant n$, and to $O(\epsilon)$ where ϵ is the scaling exponent of the eddy diffusivity in the Kraichnan model (and see below for details). In the isotropic sector ($l=0$) we recover the well known result of [2]. It is noteworthy that for higher values of l the discrete spectrum is a strictly increasing function of l. This is important, since it shows that for diminishing scales the higher order scaling exponents become irrelevant, and for sufficiently small scales only the isotropic contribution survives. As the scaling exponent appears in power laws of the type $(R/\Lambda)^{\zeta}$, with L being some typical outer scale and $R \ll \Lambda$, the larger the exponent is, the faster is the decay of the contribution as the scale R diminishes. This is precisely how the isotropization of the small scales takes place, and the higher order exponents describe the rate of isotropization. Nevertheless for intermediate scales or for finite values of the Reynolds and Peclet numbers the lower lying scaling exponents will appear in measured quantities, and understanding their role and disentangling the various contributions cannot be avoided.

The organization of this paper is as follows: In Sec. II we recall the Kraichnan model of passive scalar advection, and introduce the statistical objects of interest. In Sec. III we set up the calculation of the correlation functions of gradients of the field. It turns out that it is most straightforward to compute the fully fused correlation functions of gradient field, as these objects depend only on the ratio of the outer and inner scales. We compute these quantities and their exponents to first order in ϵ. We introduce the appropriate irreducible representations of the SO(d) symmetry group and evaluate the scaling exponents in all its sectors. In Sec. IV we turn to the correlation functions of the passive scalar field itself, and compute the scaling exponents of the structure functions in the presence of anisotropy, again correct to first order in ϵ. To this aim we compute the zero modes in all the sectors of the symmetry group. One of the interesting points of this paper is that the results of this calculation and the calculation via the correlations of the gradient fields gives the same results for the scaling exponents if one accepts the fusion rules. To clarify the issue we prove the fusion rules here in all the sectors of the symmetry group by a direct computation of the fusion of the zero modes. In Sec. V we offer a summary and a discussion.

II. KRAICHNAN'S MODEL OF TURBULENT ADVECTION AND THE STATISTICAL OBJECTS

The model of passive scalar advection with rapidly decorrelating velocity field was introduced by Kraichnan [1] already in 1968. In recent years [2–7] it was shown to be a fruitful case model for understanding multiscaling in the statistical description of turbulent fields. The basic dynamical equation in this model is for a scalar field $T(\mathbf{r},t)$ advected by a random velocity field $\mathbf{u}(\mathbf{r},t)$:

$$[\partial_t - \kappa_0\nabla^2 + \mathbf{u}(\mathbf{r},t)\cdot\boldsymbol{\nabla}]T(\mathbf{r},t) = f(\mathbf{r},t). \tag{2.1}$$

In this equation $f(\mathbf{r},t)$ is the forcing. In Kraichnan's model the advecting field $\mathbf{u}(\mathbf{r},t)$ as well as the forcing field $f(\mathbf{r},t)$ are taken to be Gaussian, time and space homogeneous, and δ correlated in time:

$$\overline{f(\mathbf{r},t)f(\mathbf{r}',t')} = \Phi(\mathbf{r}-\mathbf{r}')\,\delta(t-t'), \tag{2.2}$$

$$\langle u^{\alpha}(\mathbf{r},t)u^{\beta}(\mathbf{r}',t')\rangle = \mathcal{W}^{\alpha\beta}(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{2.3}$$

Here the symbols $\overline{\cdots}$ and $\langle\cdots\rangle$ stand for independent ensemble averages with respect to the statistics of f and $\mathbf{u}$, which are given *a priori*. We will study this model in the limit of large Peclet (Pe) number, $\mathrm{Pe} \equiv U_\Lambda \Lambda/\kappa_0$, where U_Λ is the typical size of the velocity fluctuations on the outer scale Λ of the velocity field. We stress that the forcing is not assumed isotropic, and actually the main goal of this paper is to study the statistic of the scalar field under anisotropic forcing.

The correlation function of the advecting velocity needs further discussion. It is customary to introduce $\mathcal{W}^{\alpha\beta}(\mathbf{R})$ via its $\mathbf{k}$ representation:

$$\mathcal{W}^{\alpha\beta}(\mathbf{R}) = \frac{\epsilon D}{\Omega_d}\int_{\Lambda^{-1}}^{\lambda^{-1}} \frac{d^d p}{p^{d+\epsilon}}\, P^{\alpha\beta}(\mathbf{p})\exp(-i\mathbf{p}\cdot\mathbf{R}), \tag{2.4}$$

$$P^{\alpha\beta}(\mathbf{p}) = \left[\delta_{\alpha\beta} - \frac{p^{\alpha}p^{\beta}}{p^2}\right], \tag{2.5}$$

were $P^{\alpha\beta}(\mathbf{p})$ is the transversal projector, $\Omega_d = (d-1)\Omega(d)/d$, and $\Omega(d)$ is the volume of the sphere in d dimensions [i.e., $\Omega(2)=2\pi$, $\Omega(3)=4\pi$]: Equation (2.4) introduces the four important parameters that determine the statistics of the driving velocity field: Λ and λ are the outer and inner scales of the driving velocity field, respectively. The scaling exponent ϵ characterizes the correlation functions of the velocity field, lying in the interval (0,2). The factor D is related to the correlation function (2.3) as follows:

$$\mathcal{W}^{\alpha\beta}(0) = D\,\delta_{\alpha\beta}(\Lambda^{\epsilon} - \lambda^{\epsilon}). \tag{2.6}$$

The most important property of the driving velocity field from the point of view of the scaling properties of the passive scalar is the tensor of ''eddy diffusivity'' [1]

$$\kappa_T^{\alpha\beta}(\mathbf{R}) \equiv 2[\mathcal{W}^{\alpha\beta}(0) - \mathcal{W}^{\alpha\beta}(\mathbf{R})]. \tag{2.7}$$

The scaling properties of the scalar depend sensitively on the scaling exponent ϵ that characterizes the R dependence of $\kappa_T^{\alpha\beta}(\mathbf{R})$:

$$\kappa_T^{\alpha\beta}(\mathbf{R}) \propto [\Lambda^\epsilon - \lambda^\epsilon]\delta_{\alpha\beta}, \quad \text{for } R \gg \Lambda,$$

$$\kappa_T^{\alpha\beta}(\mathbf{R}) \propto R^\epsilon \left[\delta_{\alpha\beta} - \frac{\epsilon}{d-1+\epsilon} \frac{R^\alpha R^\beta}{R^2} \right], \quad \lambda \ll R \ll \Lambda. \tag{2.8}$$

We are interested in the scaling properties of the scalar field. By this we mean the power laws characterizing the R dependence of the various correlation and response functions of $T(\mathbf{r},t)$ and its gradients. We will focus on three types of quantities:

(1) ''Unfused'' structure functions are defined as

$$S_n(\mathbf{r}_1,\overline{\mathbf{r}}_1,\ldots\mathbf{r}_n,\overline{\mathbf{r}}_n) \equiv \langle [T(\mathbf{r}_1,t) - T(\overline{\mathbf{r}}_1,t)][T(\mathbf{r}_2,t) - T(\overline{\mathbf{r}}_2,t)]\ldots[T(\mathbf{r}_n,t) - T(\overline{\mathbf{r}}_n,t)]\rangle, \tag{2.9}$$

and in particular the standard structure functions are

$$S_n(\mathbf{R}) \equiv \langle [T(\mathbf{r}+\mathbf{R},t) - T(\mathbf{r},t)]^n \rangle. \tag{2.10}$$

In writing this definition we used the fact that the stationary and space-homogeneous statistics of the velocity and the forcing fields lead to a stationary and space-homogeneous ensemble of the scalar T. If the statistics is also isotropic, then S_n becomes a function of R only, independent of the direction of $\mathbf{R}$. The ''isotropic scaling exponents'' ζ_n of the structure functions

$$S_n(R) \propto R^{\zeta_n}, \tag{2.11}$$

characterize their R dependence in the limit of large Pe, when R is in the ''inertial'' interval of scales. This range is λ, $\eta \ll R \ll \Lambda$, L, where η is the dissipative scale of the scalar field,

$$\eta = \Lambda \left(\frac{\kappa_0}{D} \right)^{1/\epsilon}. \tag{2.12}$$

(2) In addition to structure functions we are also interested in the simultaneous nth-order correlation functions of the temperature field, which is time independent in stationary statistics,

$$\mathcal{F}_n(\{\mathbf{r}_m\}) \equiv \langle T(\mathbf{r}_1,t)T(\mathbf{r}_2,t),\ldots,T(\mathbf{r}_n,t)\rangle, \tag{2.13}$$

where we used the shorthand notation $\{\mathbf{r}_m\}$ for the whole set of arguments of nth-order correlation function $\mathcal{F}_n, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$.

(3) Finally, we are interested in correlation functions of the gradient field $\boldsymbol{\nabla} T$. There can be a number of these, and we denote

$$\mathcal{H}_n^{\{\alpha_m\}}(\{\mathbf{r}_m\}) \equiv \left\langle \prod_{i=1}^n [\nabla^{\alpha_i} T(\mathbf{r}_i,t)] \right\rangle, \tag{2.14}$$

where $\{\alpha_m\}$ is a set of even n vector indices $\{\alpha_m\} = \alpha_1, \alpha_2, \ldots, \alpha_n$. We introduce also one-point correlations, which in the space homogeneous case are independent of the space coordinates:

$$H_n^{\{\alpha_m\}} \equiv \mathcal{H}_n^{\alpha_1\alpha_2\ldots\alpha_n}(\{\mathbf{r}_m = \mathbf{r}\}). \tag{2.15}$$

The tensor $\mathcal{H}_n^{\{\alpha_m\}}(\{\mathbf{r}_m\})$ can be contracted in various ways. For example, binary contractions $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$, etc. with $\mathbf{r}_1 = \mathbf{r}_2$, $\mathbf{r}_3 = \mathbf{r}_4$, etc. produce the correlation functions of the dissipation field $|\boldsymbol{\nabla} T|^2$.

When the ensemble is not isotropic we need to take into account the angular dependence of $\mathbf{R}$, and the scaling behavior consists of multiple contributions arising from anisotropic effects. The formalism to describe this is set up in Appendix A and in the forthcoming sections.

The correlation functions $\mathcal{F}_n$ satisfy the equation [1]

$$\left[-\kappa_0 \sum_{i=1}^n \nabla_i^2 + \frac{1}{2} \sum_{i,j=1}^n \kappa_T^{\alpha\beta}(\mathbf{r}_i - \mathbf{r}_j) \nabla_i^\alpha \nabla_j^\beta \right] \mathcal{F}_n(\{\mathbf{r}_m\}) = \frac{1}{2} \sum_{\{i \neq j\}=1}^n \Phi(\mathbf{r}_i - \mathbf{r}_j) \mathcal{F}_{n-2}(\{\mathbf{r}_m\}_{m \neq i,j}), \tag{2.16}$$

where $\{\mathbf{r}_m\}_{m \neq i,j}$ is the set off all $\mathbf{r}_m$ with m from 1 to n, except of $m = i$ and $m = j$. Substituting $\kappa_T^{\alpha\beta}(\mathbf{r})$ from Eqs. (2.6), (2.7) one gets

$$\left[-\kappa \sum_{i=1}^n \nabla_i^2 + \sum_{\{i \neq j\}=1}^n \mathcal{W}^{\alpha\beta}(\mathbf{r}_i - \mathbf{r}_j) \nabla_i^\alpha \nabla_j^\beta \right] \mathcal{F}_n(\{\mathbf{r}_m\}) = \frac{1}{2} \sum_{\{i \neq j\}=1}^n \Phi(\mathbf{r}_i - \mathbf{r}_j) \mathcal{F}_{n-2}(\{\mathbf{r}_m\}_{m \neq i,j}), \tag{2.17}$$

where

$$\kappa = \kappa_0 + D[\Lambda^\epsilon - \lambda^\epsilon]. \tag{2.18}$$

Here we used that in the space-homogeneous case $\Sigma_{i=1}^n \boldsymbol{\nabla}_i = 0$ and therefore

$$\left| \sum_{i=1}^n \boldsymbol{\nabla}_i \right|^2 = \sum_{i=1}^n \nabla_i^2 + \sum_{\{i \neq j\}=1}^n \nabla_i^\alpha \nabla_j^\beta = 0.$$

Consider the $\mathbf{k}$-Fourier transform of $\mathcal{F}_n$, which is defined as

$$(2\pi)^d \delta\left(\sum_{s=1}^n \mathbf{k}_s \right) F_n(\{\mathbf{k}_m\}) = \int \left[\prod_{m=1}^n d\mathbf{r}_m \exp(i\mathbf{k}_m \cdot \mathbf{r}_m) \right] \mathcal{F}_n(\{\mathbf{r}_m\}). \tag{2.19}$$

Here the δ function applies to a homogeneous ensemble in which $\mathcal{F}_n(\{\mathbf{r}_m\})$ depends only on differences of coordinates. For $F_n(\{\mathbf{k}_m\})$ Eq. (2.17) yields:

$$\kappa F_n(\{\mathbf{k}_m\})\sum_i k_i^2+\frac{\epsilon D}{\Omega d}\int_{\Lambda^{-1}}^{\lambda^{-1}}\frac{d^dp}{p^{d+\epsilon}}P^{\alpha\beta}(\mathbf{p})\times\sum_{\{i\neq j\}=1}^{n}k_i^\alpha k_j^\beta F_n(\mathbf{k}_i+\mathbf{p},\mathbf{k}_j-\mathbf{p},\{\mathbf{k}_m\}_{m\neq i,j})=\tilde{\Phi}_n(\{\mathbf{k}_m\}),\tag{2.20}$$

$$\tilde{\Phi}_n(\{\mathbf{k}_m\})\equiv\frac{(2\pi)^d}{2}\sum_{\{i\neq j\}=1}^{n}\Phi(\mathbf{k}_i)\,\delta(\mathbf{k}_i+\mathbf{k}_j)\times F_{n-2}(\{\mathbf{k}_m\}_{m\neq i,j}),\quad\text{for } n>2,\tag{2.21}$$

$$\tilde{\Phi}(\mathbf{k})=\int d\mathbf{R}\exp(i\mathbf{k}\cdot\mathbf{R})\Phi(\mathbf{R}).\tag{2.22}$$

Here $\tilde{\Phi}(k)$ is the Fourier transform of $\Phi(\mathbf{R})$ and $\tilde{\Phi}_2(\mathbf{k})=\Phi(\mathbf{k})$. Equation (2.20) may be rewritten as

$$F_n(\{\mathbf{k}_m\})=-\frac{\epsilon D}{\kappa\Omega_d}\int_{\Lambda^{-1}}^{\lambda^{-1}}\frac{d^dp}{p^{d+\epsilon}}P^{\alpha\beta}(p)\frac{\Sigma_{\{i\neq j\}=1}^{n}k_i^\alpha k_j^\beta}{\Sigma_{s=1}^{n}k_s^2}\times F_n(\mathbf{k}_i+p,\mathbf{k}_j-p,\{\mathbf{k}_m\}_{m\neq i,j})+\frac{\tilde{\Phi}_n(\{\mathbf{k}_m\})}{\kappa\Sigma_{s=1}^{n}k_s^2}.\tag{2.23}$$

This equation will serve as the basis for future analysis in Sec. III.

III. SCALING OF THE TEMPERATURE GRADIENT FIELDS

A. Basic equations in *k* representation

It appears that Eq. (2.23) is as difficult to solve as Eq. (2.16). In fact, very important information about scaling behavior may be extracted from Eq. (2.23) for small ϵ [8]. In order to develop our method we will analyze first the simultaneous, *n*-point correlation functions of the gradient fields $\mathcal{H}_n^{\{\alpha_m\}}(\{\mathbf{r}_m\})$ and $H_n^{\{\alpha_m\}}$ of Eqs. (2.14), (2.15): These objects are expressed in terms of $F_n(\{\mathbf{k}_m\})$ as follows:

$$\mathcal{H}_n^{\{\alpha_m\}}(\{\mathbf{r}_m\})=(2\pi)^{(1-n)d}\int\prod_{i=1}^{n}[d^dk_ik^{\alpha_i}\exp(i\mathbf{k}_i\cdot\mathbf{r}_i)]\times F_n(\{\mathbf{k}_m\})\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big),\tag{3.1}$$

$$H_n^{\{\alpha_m\}}=(2\pi)^{(1-n)d}\int\prod_{i=1}^{n}[d^dk_ik^{\alpha_i}]F_n(\{\mathbf{k}_m\})\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big).\tag{3.2}$$

From this and Eq. (2.23) one gets

$$H_n^{\{\alpha_m\}}=-\frac{\epsilon D}{\kappa\Omega_d}\int\frac{\Pi_{s=1}^{n}k_s^{\alpha_s}d^dk_s}{(2\pi)^{(n-1)d}}\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big)\times\int_{\Lambda^{-1}}^{\lambda^{-1}}\frac{d^dp}{p^{d+\epsilon}}\frac{P^{\alpha\beta}(\mathbf{p})\Sigma_{\{i\neq j\}=1}^{n}k_i^\alpha k_j^\beta}{\Sigma_{s=1}^{n}k_s^2}\times F_n(\mathbf{k}_i+\mathbf{p},\mathbf{k}_j-\mathbf{p},\{\mathbf{k}_m\}_{m\neq i,j})+\Psi_n^{\{\alpha_m\}},$$

$$\Psi_n^{\{\alpha_m\}}\equiv\int\frac{\Pi_{s=1}^{n}k_s^{\alpha_s}d^dk_s}{(2\pi)^{(n-1)d}}\frac{\tilde{\Phi}_n(\{\mathbf{k}_m\})}{k\Sigma_{s=1}^{n}k_s^2}\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big).\tag{3.3}$$

Shifting the dummy variables $\mathbf{k}_i-\mathbf{p}\to\mathbf{k}_i$ and $\mathbf{k}_j+\mathbf{p}\to\mathbf{k}_j$ we have another representation of this equation:

$$H_n^{\{\alpha_m\}}=-\frac{\epsilon D}{\kappa\Omega_d}\int\frac{\Pi_{s=1}^{n}d^dk_s}{(2\pi)^{(n-1)d}}\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big)\sum_{\{i\neq j\}=1}^{n}\int_{\Lambda^{-1}}^{\lambda^{-1}}\times\frac{d^dp}{p^{d+\epsilon}}\frac{(k_i^{\alpha_i}-p^{\alpha_i})(k_j^{\alpha_j}+p^{\alpha_j})P^{\alpha\beta}(\mathbf{p})k_i^\alpha k_j^\beta}{2p^2+2\mathbf{p}\cdot(\mathbf{k}_j-\mathbf{k}_i)+\Sigma_{s=1}^{n}k_s^2}\times\prod_{s=1,s\neq i,j}^{n}k_s^{\alpha_s}F_n(\{k_m\})+\Psi_n^{\{\alpha_m\}}.\tag{3.4}$$

In order to analyze this equation further we choose to nondimensionalize all the wave vectors by Λ. We write $\tilde{\mathbf{k}}_s=\Lambda\mathbf{k}_s$, $\tilde{\mathbf{p}}=\Lambda\mathbf{p}$, etc., and for simplicity drop the tilde signs at the end. We simplify the appearance of the equation further by introducing the definition of the dimensionless function

$$A_{\beta_i,\beta_j}^{\alpha_i\alpha_j}(\{k_m^2\}_{s\neq i,j}\mathbf{k}_i,\mathbf{k}_j,\mathbf{p})=-\int\frac{d\hat{\mathbf{p}}(k_i^{\alpha_i}-p^{\alpha_i})(k_j^{\alpha_j}+p^{\alpha_j})P^{\beta_i\beta_j}(\hat{\mathbf{p}})}{\Omega_d 2p^2+2\mathbf{p}\cdot(\mathbf{k}_j-\mathbf{k}_i)+\Sigma_{s=1}^{n}k_s^2},\tag{3.5}$$

where $d\hat{\mathbf{p}}$ stands for integrating over all the angles of the unit vector $\hat{\mathbf{p}}\equiv\mathbf{p}/p$. The resulting equation is

$$H_n^{\{\alpha_m\}}=g\int\frac{\Pi_{s=1}^{n}d^dk_s}{(2\pi)^{(n-1)d}}\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big)\sum_{\{i\neq j\}=1}^{n}\int_1^{\Lambda/\lambda}\times\frac{\epsilon dp}{p^{1+\epsilon}}A_{\beta_i\beta_j}^{\alpha_i\alpha_j}(\{k_m^2\}_{s\neq i,j}\mathbf{k}_i,\mathbf{k}_j,\mathbf{p})k^{\beta_i}k^{\beta_j}\times\prod_{s=1,s\neq i,j}^{n}k_s^{\alpha_s}F_n(\{k_m\})+\Psi_n^{\{\alpha_m\}},\tag{3.6}$$

where the dimensionless factor g is

$$g\equiv\frac{D\Lambda^\epsilon}{\kappa_0+D(\Lambda^\epsilon-\lambda^\epsilon)}.\tag{3.7}$$

In fact, we should recognize that the natural expansion parameter is actually not g, but $\tilde{g}$, where

$$\tilde{g}\equiv g\int_1^{\Lambda/\lambda}\frac{\epsilon dp}{p^{1+\epsilon}}.\tag{3.8}$$

Evaluating the integral we find

$$\tilde{g}=\frac{D[\Lambda^{\epsilon}-\lambda^{\epsilon}]}{\kappa_0+D(\Lambda^{\epsilon}-\lambda^{\epsilon})}. \tag{3.9}$$

We will see below that $\tilde{g}$ can take on very different values in different limiting cases. In particular it can be of $O(\epsilon)$ or of $O(1)$ depending on the order of limits. The relevant limit for the physics at hand will be discussed below. At this point we perform a calculation of $\mathcal{H}_n^{\{\alpha_m\}}$ to first order in $\tilde{g}$ in all the sectors of the symmetry group.

B. Theory to first order in $\tilde{g}$

The theory for F_n and $\mathbf{H}_n$ can be formulated iteratively, resulting in the following series:

$$F_n(\{\mathbf{r}_m\})=\sum_{q=0}^{\infty} F_{n,q}(\{\mathbf{r}_m\}),\quad H_n^{\{\alpha_m\}}=\sum_{q=0}^{\infty} H_{n,q}^{\{\alpha_m\}}. \tag{3.10}$$

Here $F_{n,q}$ is the result of the q step of the iteration procedure of Eq. (2.23), $F_{n,q}\propto\tilde{g}^q$. There are two contributions for each term of order $\tilde{g}^q$. One arises from substituting $F_{n,q-1}$ into the integral on the right-hand side (RHS) of (2.23), and the second arises from $F_{n-2,q}$, which appears in $\tilde{\Phi}_n$ according to (2.21). Correspondingly, also $H_{n,q}$ has two contributions. One is obtained from Eq. (3.6) when we substitute $F_{n,q-1}$ in the integral on the RHS, and the second when we substitute $F_{n-2,q}$ in the term denoted as Ψ_n.

In this section we compute explicitly $H_{n,1}$. Analyzing the relative importance of these two contributions to $H_{n,1}$ we found that the second contribution is negligible compared to the first when $\Lambda\ll L$. In other words, we can disregard the contribution to $H_{n,1}$ that arises from $F_{n-2,q}$. This means that for the sake of our iteration procedure we can replace $\hat{\Phi}_n$ in (2.23) by the quantity $\tilde{\Phi}_{n,0}$, which is of $O(\tilde{g}^0)$. This means that instead of Eq. (2.23) we iterate

$$F_n(\{\mathbf{k}_m\})=-\frac{\epsilon D}{\kappa\Omega_d}\int_{\Lambda^{-1}}^{\lambda^{-1}}\frac{d^dp}{p^{d+\epsilon}}P^{\alpha\beta}(\mathbf{p})\frac{\Sigma_{\{i\neq j\}=1}^{n}k_i^{\alpha}k_j^{\beta}}{\Sigma_{s=1}^{n}k_s^2}\times F_n(\mathbf{k}_i+\mathbf{p},\mathbf{k}_j-\mathbf{p},\{\mathbf{k}_m\}_{m\neq i,j})+F_{n,0}(\{\mathbf{k}_m\}), \tag{3.11}$$

$$F_{n,0}(\{\mathbf{k}_m\})=\frac{\tilde{\Phi}_{n,0}(\{\mathbf{k}_m\})}{k\Sigma_{s=1}^{n}k_s^2},$$

$$\tilde{\Phi}_{n,0}(\{\mathbf{k}_m\})\equiv\frac{(2\pi)^d}{2}\sum_{\{i\neq j\}=1}^{n}\Phi(\mathbf{k}_i)\,\delta(\mathbf{k}_i+\mathbf{k}_j)\times F_{n-2,0}(\{\mathbf{k}_m\}_{m\neq i,j}). \tag{3.12}$$

Thus we are interested in calculating

$$H_{n,1}^{\{\alpha_m\}}=\tilde{g}\int\frac{\Pi_{s=1}^{n}d^dk_s}{(2\pi)^{(n-1)d}}\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big)\sum_{\{i\neq j\}=1}^{n}\int_1^{\Lambda/\lambda}\times\frac{\epsilon dp}{p^{1+\epsilon}}A_{\beta_i\beta_j}^{\alpha_i\alpha_j}(\{k_m^2\}_{s\neq i,j}\mathbf{k}_i,\mathbf{k}_j,\mathbf{p})k^{\beta_i}k^{\beta_j}\times\prod_{s=1,s\neq i,j}^{n}k_s^{\alpha_s}F_{n,0}(\{\mathbf{k}_m\}). \tag{3.13}$$

Recall that the function $\Phi(\mathbf{k})$ is constant for $kL\ll 1$ and it vanishes for $kL\gg 1$. Therefore the leading contribution to the integrals (3.13) over $\mathbf{k}_i$ comes from the region $k_iL\leqslant 1$. In integral (3.13) $p>1/\Lambda$ and in all our approaches we consider $L\gg\Lambda$. Therefore in (3.13) $p\gg k_j$ and this equation may be simplified up to:

$$H_{n,1}^{\{\alpha_m\}}=\tilde{g}\int\frac{\Pi_{s=1}^{n}d^dk_s}{(2\pi)^{(n-1)d}}\,\delta\Big(\sum_{s=1}^{n}\mathbf{k}_s\Big)\sum_{\{i\neq j\}=1}^{n}\int_1^{\Lambda/\lambda}\frac{\epsilon dp}{p^{1+\epsilon}}\times A_{\beta_i\beta_j}^{\alpha_i\alpha_j}k^{\beta_i}k^{\beta_j}\prod_{s=1,s\neq i,j}^{n}k_s^{\alpha_s}F_{n,0}(\{\mathbf{k}_m\}), \tag{3.14}$$

where now

$$A_{\beta_i\beta_j}^{\alpha_i\alpha_j}\equiv\frac{1}{2\Omega_d}\int d\hat{\mathbf{p}}\hat{p}^{\alpha_i}\hat{p}^{\alpha_j}P_{\beta_i\beta_j}(\hat{p}), \tag{3.15}$$

is the constant tensor that obtains from the tensor function (3.5) when all $k_s\ll p$. Performing the all the wave-vector integrals we observe that the explicit ϵ is canceled by the integral over p. Accordingly

$$H_{n,1}^{\{\alpha_m\}}=\tilde{g}\sum_{\{i\neq j\}=1}^{n}A_{\beta_i\beta_j}^{\alpha_i\alpha_j}H_{n,0}^{\beta_i\beta_j\{\alpha_m\}_{m\neq i,j}}. \tag{3.16}$$

An actual integration in (3.15) yields

$$A_{\beta_i\beta_j}^{\alpha_i\alpha_j}=\frac{\delta_{\alpha_i\alpha_j}(d+1)-\delta_{\alpha_i\beta_i}\delta_{\alpha_j\beta_j}-\delta_{\alpha_i\beta_j}\delta_{\alpha_j\beta_i}}{2(d+2)(d-1)}. \tag{3.17}$$

C. Analysis in all the anisotropic sectors

We note that Eqs. (2.17) contain only isotropic operators. On the other hand $\Phi(\mathbf{r}_i-\mathbf{r}_j)$ can be anisotropic, depending on the direction of the vector $\mathbf{r}_i-\mathbf{r}_j$. Since the equations are linear, we can expand all the objects in terms of the irreducible representations of the SO(d) group of all rotations, and be guaranteed that the solutions foliate in the sense that different irreducible representations cannot be mixed. This considerations are valid for all the equations in this theory, including Eq. (3.16). To know which irreducible representations we need to use in every case one has to consult Appendix A. After doing so one notes that for any order q, the tensors $H_{n,q}^{\{\alpha_m\}}$ are constant tensors, fully symmetric in all their indices. Using the exposition of Appendix A we know that the projections on the irreducible representations of the SO(d) symmetry group must be of the form

$$H_{n,q,l}^{\{\alpha_m\}}=\lambda_q^{(l)}B_{l,\sigma,n}^{\{\alpha_m\}}. \tag{3.18}$$

Our first order calculation is aimed at finding the ratio $\lambda_1^{(l)}/\lambda_0^{(l)}$. Substituting (3.17), (3.18) in (3.16) we find

$$H_{n,1,l}^{\{\alpha_m\}} = \frac{\tilde{g}}{2(d+2)(d-1)}\Bigg[(d+1)\sum_{i\neq j}\delta^{\alpha_i\alpha_j}\delta_{\beta_i\beta_j},\lambda_0^{(l)} \times B_{l,\sigma,n}^{\beta_i\beta_j\{\alpha_m\}m\neq i,j} - \sum_{i\neq j}[\,\delta_{\beta_i}^{\alpha_i}\delta_{\beta_j}^{\beta_i}+\delta_{\beta_j}^{\alpha_i}\delta_{\beta_i}^{\alpha_j}] \times \lambda_0^{(l)}B_{l,\sigma,n}^{\beta_i\beta_j\{\alpha_m\}}\Bigg] = \frac{\tilde{g}}{(d+2)(d-1)}\left[\frac{(d+1)z_{n,l}}{2}-n(n-1)\right]\lambda_0^{(l)}B_{l,\sigma,n}^{\{\alpha_m\}}.$$

Defining now $A_n^{(l)}$ via the relation

$$\mathbf{H}_{n,1}^{(l)} = \tilde{g}A_n^{(l)}\mathbf{H}_{n,0}^{(l)}, \tag{3.19}$$

we conclude that

$$A_n^{(l)} = \frac{2n(d+n)}{d+2} - \frac{(d+1)l(l+d-2)}{2(d-1)(d+2)}. \tag{3.20}$$

D. Interpretation of the result

To interpret the result (3.19)–(3.20) we should observe that the nature of the theory that we develop depends on the order of the limits that we take. We should recognize that the quantity $H_n^{\{\alpha_m\}}$ does not exist without an inner (ultraviolet) cutoff. We are thus interested in limiting values of $\tilde{g}$ subject to the condition that η is finite. Thus one order of limits that makes sense is $\lambda\to 0$ first (corresponding to the Reynolds number going to infinity first), and then ϵ going to zero second, but keeping η fixed [for example by controlling κ_0 in Eq. (2.12)]. Another order of limits is $\epsilon\to 0$ first (still keeping η fixed, but very small) with λ being fixed and larger than η.

If we have $\lambda\to 0$ first, and then when $\epsilon\to 0$ second we find that the expansion parameter is close to unity:

$$\tilde{g}\approx 1-\left(\frac{\eta}{\Lambda}\right)^{\epsilon}, \quad \text{for} \quad \epsilon\ln\left(\frac{\eta}{\Lambda}\right)\ll 1. \tag{3.21}$$

Thus we cannot stop at (3.19), and we are forced to consider higher order terms in the expansion in $\tilde{g}$ and appropriate resummations. This is done in Sec. III F. On the other hand, if $\epsilon\to 0$ first, we find an apparently ''small'' expansion parameter that is proportional to ϵ:

$$\tilde{g}\approx \epsilon\ln\left(\frac{\Lambda}{\lambda}\right), \quad \text{for} \quad \epsilon\ln\left(\frac{\eta}{\Lambda}\right)\gg 1. \tag{3.22}$$

E. Exponentiating using renormalization group equations

Using Eq. (3.22) in Eq. (3.19) we get

$$\mathbf{H}_n^{(l)} = \left[1+\epsilon A_n^{(l)}\ln\left(\frac{\Lambda}{\lambda}\right)+O(\epsilon^2)\right]\mathbf{H}_{n,0}^{(l)}. \tag{3.23}$$

If we expect that $\mathbf{H}_n^{(l)}$ is a scale invariant function of Λ/λ we can interpret Eq. (3.23) as the beginning of an expansion that can be resummed into a power law

$$\mathbf{H}_n^{(l)} = \left(\frac{\Lambda}{\lambda}\right)^{\epsilon A_n^{(l)}}\mathbf{H}_{n,0}^{(l)}. \tag{3.24}$$

Of course, this is hardly justified just by examining the $O(\epsilon)$ term, since one can have more than one branch of scaling exponents proportional to ϵ. If we have m branches only the analysis up to $O(\epsilon^m)$ can reveal this. We return to this issue in the next subsection. An additional issue is the magnitude of λ that can be arbitrarily small, making any reexponentiation even more dubious. To overcome this problem one usually invokes the renormalization group equations to justify the exponentiation. We shortly present this method next. In doing so we want to argue that for the case in question there is nothing more in this approach than direct reexponentiation as long as higher order terms in ϵ are not included.

Within the renormalization group method [8] one considers Eq. (3.23) as the ''bare'' value of $\mathbf{H}_n^{(l)}$, $\mathbf{H}_{n,B}^{(l)}$. One then seeks a multiplicative renormalization group by defining a renormalized function

$$H_{n,R}^{(l)}(\mu,\Lambda,\ldots) = Z_H(\mu,\Lambda,\lambda)H_{n,B}^{(l)}(\Lambda,\lambda). \tag{3.25}$$

Here μ is a ''running length,'' and the only μ dependence of the RHS is through the Z_H function. Defining Z_H so that it eliminates the dependence of the left-hand side (LHS) on λ and setting the initial condition $\mathbf{H}_{n,R}(\Lambda,\Lambda,\ldots)=\mathbf{H}_{n,0}^{(l)}$ we get

$$Z_H(\mu,\lambda) = 1+\epsilon A_n^{(l)}\ln(\lambda/\mu). \tag{3.26}$$

From Eq. (3.25) we get

$$\frac{d\ln[H_{n,R}^{(l)}(\mu,\Lambda,\ldots)]}{d\ln\mu} = \gamma_H, \tag{3.27}$$

where

$$\gamma_H = -\epsilon A_n^{(l)}+O(\epsilon^2). \tag{3.28}$$

Solving the differential equation (3.27) we get

$$\mathbf{H}_{n,R}^{(l)}(\mu,\Lambda,\ldots) = \left(\frac{\Lambda}{\mu}\right)^{\epsilon A_n^{(l)}}\mathbf{H}_{n,0}. \tag{3.29}$$

Exponentiating (3.26) and solving Eq. (3.25) in favor of $\mathbf{H}_{n,B}^{(l)}$ we recover Eq. (3.24). Note that the inner scale in this case is λ, since $\lambda>\eta$. This fact casts an additional doubt on this limit of the theory, since it misses altogether the existence of the Betchelor regime [9] between λ and η. For all these reasons we tend to disqualify (3.24) despite the relative simplicity of its derivation. We turn next to the other limiting procedure.

F. Theory for $\lambda\to 0$ first

Considering the limit $\lambda\to 0$ first, Eq. (3.19) is still valid, but now $\tilde{g}$ is of order unity, and we cannot justify reexponentiation. The small parameter ϵ seems to have disappeared.

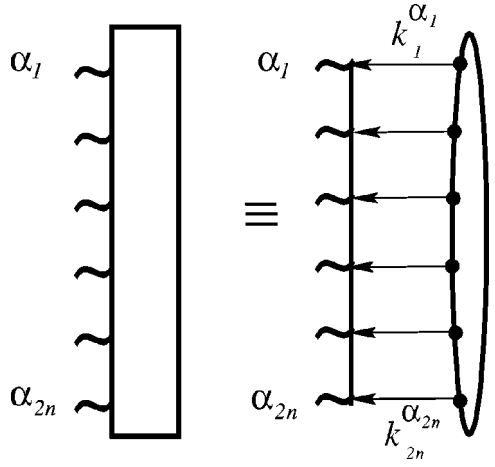

FIG. 1. The graphic representation of Eq. (3.3).

This forces us to consider all the higher order terms in $\tilde{g}$ to understand how to resum them. We will see that at the end ϵ reappears.

The resummation of the $\tilde{g}$ dependence is aided significantly by some graphic representations of the relevant equations and their perturbative solutions. In Fig. 1 we represent graphically the definition (3.2) of $H_n^{\{\alpha_m\}}$ in terms of $F_n(\{\mathbf{k}_m\})$. This helps us to introduce the basic diagrammatic notations.

A solid long rectangle with $n\alpha_m$ wavy lines stands for $H_n^{\{\alpha_m\}}$, while the elongated solid ellipse represents F_n. The $\mathbf{k}_s$ wave vectors are denoted by arrows, and the dots represent multiplications. The vertical line connecting all the arrow heads stands for the integration with a δ function over the sum of all the $\mathbf{k}$ vectors. Figure 2 is a graphic notation of Eq. (3.6). The little ellipse with a vertical arrow designated by $\mathbf{p}$ stands for the tensor function $A^{\alpha_i\alpha_j}_{\beta_i\beta_j}(\{\mathbf{k}_s\}_{s\neq i,j},\mathbf{k}_i,\mathbf{k}_j,\mathbf{p})$. This ellipse is involved in the integration over the vector $\mathbf{p}$ in the loop to the left of the ellipse with a weight consisting of the sum of squares of the $\mathbf{k}$ vectors.

The thin elongated ellipse in the second term on the RHS stands for the zeroth order term $F_{n,0}$, cf., Eq. (3.12). The actual values of the wave vectors are indicated in this diagram. In later diagrams, Fig. 3, we drop this obvious notation. In Fig. 3 we display the perturbative solution, which results from the iteration procedure in Eq. (3.11), the result of which is substituted in Eq. (3.3).

The first diagram on the RHS is $\mathbf{H}^{(l)}_{n,0}$, whereas the second is the first order term $\mathbf{H}^{(l)}_{n,1}$, Eq. (3.19). The last diagram shown in $\mathbf{H}^{(l)}_{n,2}$. One should note that when the $A^{\alpha_i\alpha_j}_{\beta_i\beta_j}$ ellipse

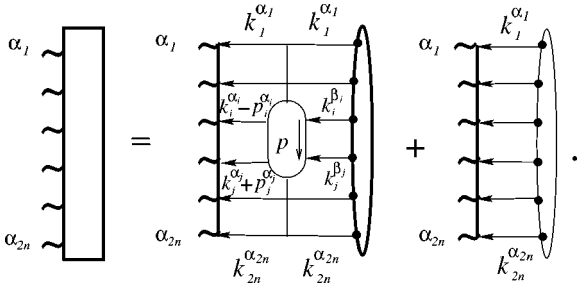

FIG. 2. The graphic representation of Eq. (3.8).

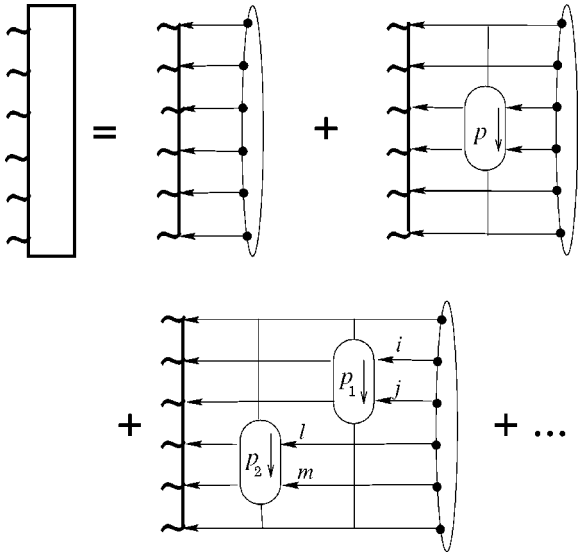

FIG. 3. The graphic representation of the iteration scheme.

appears once we have a sum over all pairs i,j indices with $i\neq j$. When it appears twice there is a double sum, with respect to the pairs $i\neq j$ and $l\neq m$. In analyzing such diagrams one needs to identify three distinct possibilities. These are denoted as case (a), $i=l$, $j=m$, case (b) $i=l$ but $j\neq m$, and case (c) where all the indices are different. There are two integrals over $\mathbf{p}_1$ and $\mathbf{p}_2$ in the loops to the left of the corresponding ellipse. We refer to the functions $\mathbf{A}$ of Eq. (3.5) that appear in these integrals as $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Analyzing the integrals it is useful to separate the discussion to region I, in which $p_1>p_2$, and region II, in which $p_2>p_1$. In region I we can neglect the contribution of $\mathbf{p}_2$ and all $\mathbf{k}_s$ with respect to $\mathbf{p}_1$. Accordingly we have two independent integrals and sums, and the result is therefore

$$\mathbf{H}^{(l)}_{n,2}=\frac{1}{2}[\tilde{g}A^{(l)}_n]^2\mathbf{H}^{(l)}_{n,0}\ \text{(region I)}, \tag{3.30}$$

where the factor 1/2 stems from the fact that the volume of region I is a half of the whole volume of $(\mathbf{p}_1,\mathbf{p}_2)$ space. In region II we should distinguish between the cases (a), (b), and (c), for which the evaluation of $\mathbf{A}_2$ will be different. In case (c) Eq. (3.5) shows that $\mathbf{A}_2$ is of the order of p_2^2/p_1^2, which is small. In case (b) $\mathbf{A}_2$ is of the order of p_2/p_1, which is still small. Only case (a), in which the loops appear as two rungs on the same ladder, we have $\mathbf{A}_2$ of the order of unity. The actual calculation of this integral is presented in Appendix B, with the final result

$$\mathbf{H}^{(l)}_{n,2}=\frac{1}{2}\tilde{g}^2A^{(l)}_n\mathbf{H}^{(l)}_{n,0}\quad[\text{region II, case (a)}]. \tag{3.31}$$

Together the second order result for $\mathbf{H}^{(l)}_{n,2}$ is

$$\mathbf{H}^{(l)}_{n,2}=\frac{1}{2}\tilde{g}^2[A^{(l)}_n+(A^{(l)}_n)^2]\mathbf{H}^{(l)}_{n,0}. \tag{3.32}$$

Our aim is to find the fully resummed form, correct to all order in $\tilde{g}$ and $A^{(l)}_n$, of $\mathbf{H}^{(l)}_n$. We can express it in the form

$$\mathbf{H}_n^{(l)} \equiv K(\tilde{g},A_n^{(l)})\mathbf{H}_{n,0}^{(l)}, \tag{3.33}$$

where the function $K(\tilde{g},A_n^{(l)})$ is represented as the double infinite sum

$$K(\tilde{g},A) = 1 + \sum_{m=1}^{\infty} A^m \sum_{s=m}^{\infty} D_{m,s}\tilde{g}^s. \tag{3.34}$$

Up to now we have information about $D_{1,1}=1$ and $D_{1,2}=D_{2,2}=1/2$.

In Appendix C we derive the following recurrent relation for the higher order terms

$$D_{1,s}=\frac{1}{s},\quad D_{m+1,s}=\frac{1}{s}\sum_{q=m}^{s-1} D_{q,m}. \tag{3.35}$$

Using (3.35) in Eq. (3.34) we find the contribution proportional to A:

$$K_1(\tilde{g},A) = A\sum_{s=1}^{\infty}\frac{\tilde{g}^s}{s} = -A\ln(1-\tilde{g}). \tag{3.36}$$

Considering all the terms quadratic in A, and using the recurrent relations to determine $D_{2,s}$ we find

$$K_2(\tilde{g},A) = A^2\sum_{s=2}^{\infty}\frac{\tilde{g}^s}{s}\sum_{q=1}^{s-1}\frac{1}{q}. \tag{3.37}$$

This double sum is computed in Appendix C with the result

$$K_2(\tilde{g},A) = \frac{1}{2}[-A\ln(1-\tilde{g})]^2. \tag{3.38}$$

The general result can be derived using similar techniques with the result

$$K_m(\tilde{g},A) = \frac{1}{m!}[-A\ln(1-\tilde{g})]^m. \tag{3.39}$$

Accordingly we conclude with the series for $K(\tilde{g},A)$:

$$K(\tilde{g},A) = \sum_{m=0}^{\infty}\frac{[-A\ln(1-\tilde{g})]^m}{m!} = \exp[-A\ln(1-\tilde{g})]. \tag{3.40}$$

Using now Eq. (3.21) one finds

$$K(\tilde{g},A) = \left(\frac{\Lambda}{\eta}\right)^{\epsilon A}. \tag{3.41}$$

Finally, using Eq. (3.33) we have the final result

$$\mathbf{H}_n^{(l)} = \left(\frac{\Lambda}{\eta}\right)^{\epsilon A_n^{(l)}}\mathbf{H}_{n,0}^{(l)}. \tag{3.42}$$

We are pleased to find that the inner scale is now η, in agreement with our expectation. The exponentiation was achieved naturally in the present case.

In assessing this result, we need to return to a delicate point in the derivation of Eq. (3.42). The procedure involved summing all the terms of the order of unity (powers of $\tilde{g}\approx 1$), while neglecting terms of $O(\epsilon)$. However, the sums (3.38)–(3.39) result in expressions containing $(\tilde{g}-1)\propto\epsilon$. In other words, we end up with terms that appear of the same order as those neglected during the procedure. In order to justify the results (3.42) we must go back and analyze contributions of $O(\epsilon)$. These appear, for example, in contributions in which the ''rungs'' appear on adjacent ladders, like case (b) in region II. The two ''simple'' rungs appearing in two adjacent ladders can be now considered as a single compounded rung. We focus now on the infinite set of diagrams in which this compounded rung repeats many times. The sum of such diagrams will again generate results containing $(\tilde{g}-1)$, and in the end will be responsible for terms of $O(\epsilon^2)$ in the scaling exponent.

The reason for this phenomenon is the structure of the iterative solution. Sums of terms of order $\tilde{g}$ have cancellations, leading eventually to a result of $O(\epsilon)$. The sum of terms of $O(\epsilon)$ have a very similar structure, just we a redefined ''rung.'' Therefore they automatically generate another factor of ϵ by resummation. This phenomenon repeats in higher orders, again by redefining what do mean by a ''rung.''

Thus Eq. (3.42) can be considered as the final result for the scaling of the fused correlation function of gradient fields with the exponent correct to $O(\epsilon)$. In the next section we turn to the calculation of the scaling exponent of the unfused correlation function of the scalar field itself. We will show that the exponents computed in both methods agree when the same objects are evaluated. This agreement is connected in the final section with the existence of fusion rules that control the asymptotic properties of unfused correlation functions when some coordinates are fused together.

IV. ZERO MODES IN THE ANISOTROPIC SECTORS

A. Calculation of the correlation functions

In this section we consider the zero modes of Eq. (2.16). In other words we seek solutions $Z_n(\{\mathbf{r}_m\})$ that in the inertial interval solve the homogeneous equation

$$\sum_{i\neq j=1}^{n}\kappa_T^{\alpha\beta}(\mathbf{r}_i-\mathbf{r}_j)\nabla_i^{\alpha}\nabla_j^{\beta}Z_n(\{\mathbf{r}_m\}) = 0. \tag{4.1}$$

We allow anisotropy on the large scales. Since all the operators here are isotropic and the equation is linear, the solution space foliates into sectors $\{l,\sigma\}$ corresponding to the irreducible representations of the SO(d) symmetry group. Accordingly we write the wanted solution in the form

$$Z_n(\{\mathbf{r}_m\}) = \sum_{l,\sigma} Z_{n,l,\sigma}(\{\mathbf{r}_m\}), \tag{4.2}$$

where $Z_{n,l,\sigma}$ is composed of functions that transform according to the (l,σ) irreducible representations of SO(d). Each of these components is now expanded in ϵ. In other words, we write, in the notation of Ref. [2],

$$Z_{n,l,\sigma} = E_{n,l,\sigma} + \epsilon G_{n,l,\sigma} + O(\epsilon^2). \tag{4.3}$$

For $\epsilon=0$, Eq. (4.1) simplifies to

$$\sum_{i=1}^{n} \nabla_i^2 E_{n,l,\sigma}(\{\mathbf{r}_m\}) = 0, \tag{4.4}$$

for any value of l, σ. Next we expand the operator in Eq. (4.1) in ϵ and collect the terms of $O(\epsilon)$:

$$\sum_{i=1}^{n} \nabla_i^2 G_{n,l,\sigma}(\{\mathbf{r}_m\}) = V_n E_{n,l,\sigma}(\{\mathbf{r}_m\}), \tag{4.5}$$

where ϵV_n is the first order term in the expansion of the operator in (4.1),

$$V_n \equiv \sum_{j\neq k=1}^{n} \left[\delta^{\alpha\beta} \ln(r_{jk}) - \frac{r_{jk}^{\alpha} r_{jk}^{\beta}}{(d-1) r_{jk}^2} \right] \nabla_j^{\alpha} \nabla_k^{\beta}, \tag{4.6}$$

where $\mathbf{r}_{jk} \equiv \mathbf{r}_j - \mathbf{r}_k$.

In solving Eq. (4.4) we are led by the following considerations: we want scale invariant solutions, which are powers of $\mathbf{r}_{jk}$. We want analytic solutions, and thus we are limited to polynomials. Finally we want solutions that involve all the n coordinates for the function $E_{n,l,\sigma}$; solutions with fewer coordinates do not contribute to the structure functions (2.9). To see this, note that the unfused structure function is a linear combination of correlation functions. This linear combination can be represented in terms of the difference operator $\delta_j(\mathbf{r},\mathbf{r}')$ defined by

$$\delta_j(\mathbf{r},\mathbf{r}')\mathcal{F}(\{\mathbf{r}_m\}) \equiv \mathcal{F}(\{\mathbf{r}_m\})\big|_{\mathbf{r}_j=\mathbf{r}} - \mathcal{F}(\{\mathbf{r}_m\})\big|_{\mathbf{r}_j=\mathbf{r}'}. \tag{4.7}$$

Then,

$$\mathcal{S}_n(\mathbf{r}_1,\mathbf{r}_1',\ldots,\mathbf{r}_n\mathbf{r}_n') = \prod_j \delta_j(\mathbf{r}_j,\mathbf{r}_j')\mathcal{F}(\{\mathbf{r}_m\}). \tag{4.8}$$

Accordingly, if $\mathcal{F}(\{\mathbf{r}_m\})$ does not depend on $\mathbf{r}_k$, then $\delta_k(\mathbf{r}_k,\mathbf{r}_k')\mathcal{F}(\{\mathbf{r}_m\}) = 0$ identically. Since the difference operators commute, we can have no contribution to the structure functions from parts of $\mathcal{F}$ that depend on less than n coordinates. Finally we want the minimal polynomial because higher order ones are negligible in the limit $r_{jk} \ll \Lambda$. Accordingly, $E_{n,l,\sigma}$ with $l \leqslant n$ is a polynomial of order n. Consulting Appendix A for the irreducible representations of the SO(d) symmetry group, we can write the most general form of $E_{n,l,\sigma}$, up to an arbitrary factor, as

$$E_{n,l,\sigma} = r_1^{\alpha_1} \ldots r_n^{\alpha_n} B_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n} + [\cdots], \tag{4.9}$$

where $[\cdots]$ stands for all the terms that contain less than n coordinates; these do not appear in the structure functions, but maintain the translational invariance of our quantities. The appearance of the tensor $B_{n,l,\sigma}^{\alpha_1\ldots\alpha_n}$ of Appendix A is justified by the fact that $E_{n,l,\sigma}$ must be symmetric to permutations of any pair of coordinates on the one hand, and it has to belong to the l, σ sector on the other hand. This requires the appearance of the fully symmetric tensor (A5).

In light of Eqs. (4.5)–(4.6) we seek solution for $G_n^{(l)}(\{\mathbf{r}_m\})$ of the form

$$G_{n,l\sigma}(\{\mathbf{r}_m\}) = \sum_{j\neq k} H_{l,\sigma}^{jk}(\{\mathbf{r}_m\}) \ln(r_{jk}) + H_{l,\sigma}(\{\mathbf{r}_m\}), \tag{4.10}$$

where $H_{l,\sigma}^{jk}(\{\mathbf{r}_m\})$ and $H_{l,\sigma}(\{\mathbf{r}_m\})$ are polynomials of degree n. The latter is fully symmetric in the coordinates. The former is symmetric in r_j, r_k and separately in all the other $\{r_m\}_{m\neq i,j}$.

Substituting Eq. (4.10) into Eq. (4.5) and collecting terms of the same type yields three equations:

$$\sum_i \nabla_i^2 H_{l,\sigma}^{jk} = \nabla_j \cdot \nabla_k E_{2n,l,\sigma}, \tag{4.11}$$

$$2[d-2+\mathbf{r}_{jk}\cdot(\boldsymbol{\nabla}_j - \boldsymbol{\nabla}_k)]H_{l,\sigma}^{jk} + \frac{r_{jk}^{\alpha} r_{jk}^{\beta} \nabla_j^{\alpha} \nabla_k^{\beta}}{d-1} E_{2n,l,\sigma} = -r_{jk}^2 K_{l,\sigma}^{jk}, \tag{4.12}$$

$$\sum_i \nabla_i^2 H_{l,\sigma} = \sum_{j\neq k} K_{l,\sigma}^{jk}. \tag{4.13}$$

Here $K_{l,\sigma}^{jk}$ are polynomials of degree $n-2$, which are separately symmetric in the j,k coordinates and in all the other coordinates except j,k. In Ref. [2] it was proven that for $l=0$ these equations posses a unique solution. The proof follows through unchanged for any $l \neq 0$, and we thus proceed to finding the solution.

By symmetry we can specialize the discussion to $j=1, k=2$. In light of Eq. (4.12) we see that $H_{l,\sigma}^{12}$ must have at least a quadratic contribution in r_{12}. This guarantees that (4.10) is nonsingular in the limit $r_{12} \to 0$. The only part of $H_{l,\sigma}^{12}$ that will contribute to structure functions must contain $\mathbf{r}_3 \cdots \mathbf{r}_n$ at least once. Since $H_{l,\sigma}^{12}$ has to be a polynomial of degree n in the coordinates, it must be of the form

$$H_{l,\sigma}^{12} = r_{12}^{\alpha_1} r_{12}^{\alpha_2} r_3^{\alpha_3} \cdots r_n^{\alpha_n} C^{\alpha_1\alpha_2,\ldots,\alpha_n} + [\cdots]_{1,2}, \tag{4.14}$$

where $[\cdots]_{1,2}$ contains terms with higher powers of r_{12} and therefore do not contain some of the other coordinates $r_3 \cdots r_n$. Obviously such terms are unimportant for the structure functions. Since $H_{l,\sigma}^{12}$ has to be symmetric in $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3 \cdots \mathbf{r}_n$ separately, and it has to belong to an l, σ sector, we conclude that the constant tensor $\mathbf{C}$ must have the same symmetry and to belong to the same sector. Consulting Appendix A, the most general form of $\mathbf{C}$ is

$$C^{\alpha_1\alpha_2\ldots\alpha_n} = a B_{2n,l,\sigma}^{\alpha_1,\alpha_2,\ldots,\alpha_n} + b\,\delta^{\alpha_1\alpha_2} B_{2n-2,l,\sigma}^{\alpha_3,\alpha_4,\ldots,\alpha_n} + c \sum_{i\neq j>2} \delta^{\alpha_1\alpha_i}\delta^{\alpha_2\alpha_j} B_{2n-4,l,\sigma}^{\alpha_3,\alpha_4,\ldots,\alpha_n}. \tag{4.15}$$

Substituting in Eq. (4.12) one finds

$$(d+2)H_{l,\sigma}^{12} + \frac{r_{12}^{\alpha_1} r_{12}^{\alpha_2} r_3^{\alpha_3} \ldots r_{2n}^{\alpha_n}}{2d-2} B_{2n,l,\sigma}^{\alpha_1,\ldots,\alpha_n} + \frac{1}{2} r_{12}^{\alpha_1} r_{12}^{\alpha_2} \delta^{\alpha_1\alpha_2} K_{l,\sigma}^{1,2} = [\cdots]_{1,2}. \tag{4.16}$$

Substituting Eq. (4.14) and demanding that coefficients of the term $r_1^{\alpha_1}\ldots r_n^{\alpha_n}$ will sum up to zero, we obtain

$$-2(d+2)a-\frac{2}{2d-2}=0,$$

$$-2(d+2)c=0;\ \Rightarrow c=0. \tag{4.17}$$

The coefficient b is not determined from this equation due to possible contributions from the unknown last term. We determine the coefficient b from Eq. (4.11). After substituting the forms we find

$$4\delta^{\alpha_1\alpha_2}r_3^{\alpha_3}\cdots r_{2n}^{\alpha_n}[aB^{\alpha_1,\ldots,\alpha_n}_{2n,l,\sigma}+b\,\delta^{\alpha_1\alpha_2}B^{\alpha_3,\alpha_4,\ldots,\alpha_n}_{2n-2,l,\sigma}]$$
$$=\delta^{\alpha_1\alpha_2}r^{\alpha_3}\cdots r^{\alpha_n}B^{\alpha_1,\ldots,\alpha_n}_{2n,l,\sigma}+[\cdots]_{1,2}. \tag{4.18}$$

Recalling the identity (A6) we obtain

$$b=\frac{z_{n,l}}{4d}[1-4a]. \tag{4.19}$$

Finally we find that a is n,l independent,

$$a=-\frac{1}{2(d+2)(d-1)}, \tag{4.20}$$

whereas b does depend on n and l, and we therefore denote it as $b_{n,l}$

$$b_{n,l}=\frac{(d+1)}{4(d+2)(d-1)}z_{n,l}. \tag{4.21}$$

In the next subsection we compute from these results the scaling exponents in the sectors of the SO(d) symmetry group with $l\leqslant n$.

B. The scaling exponents of the structure functions

We now wish to show that the solution for the zero modes of the correlation functions $\mathcal{F}_n$ (i.e., Z_n) result in homogeneous structure functions $\mathcal{S}_n$. In every sector $l\leqslant n$, σ we compute the scaling exponents, and show that they are independent of σ. Accordingly the scaling exponents are denoted ζ_n^l, and we compute them to first order in ϵ.

Using (4.7), (4.8), the structure function is given by

$$S_{n,l,\sigma}(\mathbf{r}_n,\bar{\mathbf{r}}_1;\ldots;\mathbf{r}_n,\bar{\mathbf{r}}_n)$$
$$=\Delta_1^{\alpha_1}\ldots\Delta_n^{\alpha_n}B^{\alpha_1,\ldots,\alpha_n}_{n,l,\sigma}$$
$$+\epsilon\sum_{i\neq j}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)$$
$$\times[aB^{\alpha_1,\ldots,\alpha_n}_{n,l,\sigma}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B^{\alpha_1,\ldots,\alpha_n}_{n-2,l,\sigma}}^{\text{no } i,j}], \tag{4.22}$$

where $\Delta_i^{\alpha_i}\equiv r_i^{\alpha_i}-\bar{r}_i^{\alpha_i}$, and the function f is defined as:

$$f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)$$
$$\equiv(r_i-r_j)^{\alpha_i}(r_i-r_j)^{\alpha_j}\ln|\mathbf{r}_i-\mathbf{r}_j|+(\bar{r}_i-\bar{r}_j)^{\alpha_i}$$
$$\times(\bar{r}_i-\bar{r}_j)^{\alpha_j}\ln|\bar{\mathbf{r}}_i-\bar{\mathbf{r}}_j|-(r_i-\bar{r}_j)^{\alpha_i}(r_i-\bar{r}_j)^{\alpha_i}$$
$$\times\ln|\mathbf{r}_i-\bar{\mathbf{r}}_j-(\bar{r}_i-r_j)^{\alpha_i}(\bar{r}_i-r_j)^{\alpha_j}\ln|\bar{\mathbf{r}}_i-\mathbf{r}_j|. \tag{4.23}$$

The scaling exponent of $\mathcal{S}_{n,l,\sigma}$ can be found by multiplying all its coordinates by μ. A direct calculation yields:

$$\mathcal{S}_{n,l,\sigma}(\mu\mathbf{r}_1,\mu\bar{\mathbf{r}}_1;\ldots)$$
$$=\mu^n\mathcal{S}_{n,l,\sigma}(\mathbf{r}_1,\bar{\mathbf{r}}_1;\ldots)-2\epsilon\mu^n\ln\mu$$
$$\times\sum_{i\neq j}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j}\Delta_i^{\alpha_i}\Delta_j^{\alpha_j}$$
$$\times[aB^{\alpha_1,\ldots,\alpha_n}_{n,l,\sigma}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B^{\alpha_1,\ldots,\alpha_n}_{n-2,l,\sigma}}^{\text{no } i,j}]+O(\epsilon^2),$$
$$=\mu^n\mathcal{S}_{n,l,\sigma}(\mathbf{r}_1,\bar{\mathbf{r}}_1;\ldots)-2\epsilon\mu^n\ln\mu\Delta_1^{\alpha_1}\ldots\Delta_n^{\alpha_n}$$
$$\times\sum_{i\neq j}[aB^{\alpha_1,\ldots,\alpha_n}_{n,l,\sigma}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B^{\alpha_1,\ldots,\alpha_n}_{n-2,l,\sigma}}^{\text{no } i,j}]$$
$$+O(\epsilon^2).$$

Using (A8), we find that

$$\sum_{i\neq j}[aB^{\alpha_1,\ldots,\alpha_n}_{n,l,\sigma}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B^{\alpha_1,\ldots,\alpha_n}_{n-2,l,\sigma}}^{\text{no } i,j}]$$
$$=[n(n-1)a+b_{n,l}]B^{\alpha_1,\ldots,\alpha_n}_{n,l,\sigma},$$

and therefore, we finally obtain

$$\mathcal{S}_n(\mu\mathbf{r}_1,\mu\bar{\mathbf{r}}_1;\ldots)=\mu^n\{1-2\epsilon[n(n-1)a+b_{n,l}]\ln\mu\}$$
$$\times\mathcal{S}_n(\mathbf{r}_1,\bar{\mathbf{r}}_1;\ldots)+O(\epsilon)^2$$
$$=\mu^{\zeta_n^{(l)}}\mathcal{S}_n(\mathbf{r}_1,\bar{\mathbf{r}}_1;\ldots)+O(\epsilon^2).$$

The result of the scaling exponent is now evident:

$$\zeta_n^{(l)}=n-2\epsilon\left[-\frac{n(n-1)}{2(d+2)(d-1)}+\frac{(d+1)}{4(d+2)(d-1)}z_{n,l}\right]$$
$$+O(\epsilon^2)$$
$$=n-\epsilon\left[\frac{n(n+d)}{2(d+2)}-\frac{(d+1)l(l+d-2)}{2(d+2)(d-1)}\right]$$
$$+O(\epsilon^2). \tag{4.24}$$

For $l=0$ this result coincides with [2]. This is the final result of this calculation. It is noteworthy that this result is in full agreement with (3.42) and (3.20), even though the scaling exponents that appear in these results refer to different quantities. The way to understand this is the fusion rules that are discussed next.

C. Fusion rules

The fusion rules address the asymptotic properties of the fully unfused structure functions when two or more of the coordinates are approaching each other, whereas the rest of the coordinates remain separated by much larger scales. A full discussion of the fusion rules for the Navier-Stokes and the Kraichnan model can be found in [7,10]. In this section we wish to derive the fusion rules directly from the zero modes that were computed to $O(\epsilon)$, in all the sectors of the symmetry group. In other words, we are after the dependence of the structure function $\mathcal{S}_n(\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots)$ on its first p pairs of coordinates $\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots;\mathbf{r}_p,\overline{\mathbf{r}}_p$ in the case where these points are very close to each other compared to their distance from the other $n-p$ pairs of coordinates. Explicitly, we consider the case where $\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots;\mathbf{r}_p,\overline{\mathbf{r}}_p \ll \mathbf{r}_{p+1},\overline{\mathbf{r}}_{p+1};\ldots;\mathbf{r}_n,\overline{\mathbf{r}}_n$. (We have used here the property of translational invariance to put the center of mass of the first $2p$ coordinates at the origin.) The calculation is presented in Appendix D, with the final result [to $O(\epsilon)$]

$$\mathcal{S}_{n,l,\sigma}(\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots;\mathbf{r}_n,\overline{\mathbf{r}}_n) = \sum_{j=j_{\max}}^{p} \sum_{\sigma'} \boldsymbol{\psi}_{j,\sigma'} S_{p,j,\sigma'}(\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots;\mathbf{r}_p\overline{\mathbf{r}}_p). \tag{4.25}$$

In this expression the quantity $\psi_{j,\sigma'}$ is a function of all the coordinates that remain separated by large distances, and

$$j_{\max}=\max\{0,p+l-n\}, \quad l\leqslant n. \tag{4.26}$$

We have shown that the LHS has a homogeneity exponent $\zeta_n^{(l)}$. The RHS is a product of functions with homogeneity exponents $\zeta_p^{(j)}$ and the functions $\psi_{j,\sigma'}$. Using the linear independence of the functions $\boldsymbol{\mathcal{S}}_{p,j,\sigma'}$ we conclude that $\psi_{j,\sigma'}$ must have homogeneity exponent $\zeta_n^{(l)}-\zeta_p^{(j)}$. This is precisely the prediction of the fusion rules, but in each sector separately. One should stress the intuitive meaning of the fusion rules. The result shows that when p coordinates approach each other, the homogeneity exponent corresponding to these coordinates becomes simply $\zeta_p^{(j)}$ as if we were considering a p-order correlation function. The meaning of this result is that p field amplitudes measured at p close by coordinates in the presence of $n-p$ field amplitudes determined far away behave scalingwise, like p field amplitudes in the presence of anisotropic boundary conditions. In closing, we note that the tensor functions $\boldsymbol{\psi}_{j,\sigma'}$ do not necessarily belong to the j, σ' sector of SO(d).

V. SUMMARY AND DISCUSSION

One motivation in this paper was to understand the scaling properties of the statistical objects under anisotropic boundary conditions. The scaling exponents were found for all $l\leqslant n$, cf., Eq. (1.1). We found a discrete and strictly increasing spectrum of exponents as a function of l. This means that for higher l the anisotropic contributions to the statistical objects decay faster upon decreasing scales. In other words, the statistical objects tend towards locally isotropic statistics upon decreasing the scale. The rate of isotropization is determined by the difference between the l-dependent scaling exponents, and is of course a power law. The result shows that the l-dependent part is n independent. This means that the rate of isotropization of all the moments of the distribution function of field differences across a given scale is the same. This is a demonstration of the fact that the distribution function itself tends towards a locally isotropic distribution function at the same rate. We note in passing that to first order in ϵ the l-dependent part is also identical for ζ_2, a quantity whose isotropic value is not anomalous. For all $l>1$ also $\zeta_2^{(l)}$ is anomalous, and in agreement with the $n=1$ value of Eq. (1.1). Significantly, for ζ_2 we have a nonperturbative result that was derived in [7], namely,

$$\zeta_2^{(l)}=\frac{1}{2}\left[2-d-\epsilon + \sqrt{(2-d-\epsilon)^2+\frac{4(d+\epsilon-1)l(d+l-2)}{d-1}}\right], \tag{5.1}$$

valid for all values of ϵ in the interval (0,2) and for all $l\geqslant 2$. This exact result agrees after expanding to $O(\epsilon)$ with (4.24) for $n=1$ and $l=2$.

Our second motivation was to expose the correspondence between the scaling exponents of the zero modes in the inertial interval and the corresponding scaling exponents of the gradient fields. The latter do not depend on any inertial scales, and the exponent appears in the combination $(\Lambda/\xi)^{\zeta_n^l}$, where ξ is the appropriate ultraviolet inner cutoff, either λ or η, depending on the limiting process. We found exact agreement with the exponents of the zero modes in all the sectors of the symmetry group and for all values of n. The reason behind this agreement is the linearity of the fundamental equation of the passive scalar (2.1). This translates to the fact that the viscous cutoff η, Eq. (2.12), is n and l independent, and also does not depend on the inertial separations in the unfused correlation functions. This point has been discussed in detail in [7,11]. In the case of Navier-Stokes statistics we expect this ''trivial'' correspondence to fail, but nevertheless the ''bridge relations'' that connect these two families of exponents has been presented in [10] for the isotropic sector. Finally we note that in the present case we have displayed the fusion rules in all the l sectors, using the $O(\epsilon)$ explicit form of the zero modes. We expect the fusion rules to have a nonperturbative validity for any value of ϵ. It would be interesting to explore similar results for the Navier-Stokes case.

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APPENDIX A: ANISOTROPY IN *d* DIMENSIONS

To deal with anisotropy in d dimensions we need classify the irreducible representations of the group of all d-dimensional rotations, SO(d) [12], and then to find a proper basis for these representations. The main linear space that we work in (the carrier space) is the space of constant tensors with n indices. This space possesses a natural representation of SO(d), given by the well known transformation of tensors under d-dimensional rotation.

The traditional method to find a basis for the irreducible representations of SO(d) in this space is using the Young tableaux machinery on the subspace of traceless tensors [12]. It turns out that in the context of the present paper, we do not need the explicit structure of these tensors. Instead, all that matters are some relations among them. A convenient way to derive these relations is to construct the basis tensors from functions on the unit d-dimensional sphere that belong to a specific irreducible representation. Here also, the explicit form of these functions in unimportant. All that matters for our calculations is the action of the Laplacian operator on these functions.

Let us therefore consider first the space $\mathcal{S}_d$ of functions over the unit d-dimensional sphere. The representation of SO(d) over this space is naturally defined by

$$\mathcal{O}_{\mathcal{R}}\Psi(\hat{u}) \equiv \Psi(\mathcal{R}^{-1}\hat{u}), \tag{A1}$$

where $\Psi(\hat{u})$ is any function on the d-dimensional sphere, and $\mathcal{R}$ is a d-dimensional rotation.

$\mathcal{S}_d$ can be spanned by polynomials of the unit vector $\hat{u}$. Obviously (A1) does not change the degree of a polynomial, and therefore each irreducible representation in this space can be characterized by an integer $l=0, 1, 2,...$, specifying the degree of the polynomials that span this representation. At this point, we cannot rule out the possibility that some other integers are needed to fully specify all irreducible representations in $\mathcal{S}_d$ and therefore we will need below another set of indices to complete the specification.

We can now choose a basis of polynomials $\{Y_{l,\sigma}(\hat{u})\}$ that span all the irreducible representations of SO(d) over $\mathcal{S}_d$. The index σ counts all integers other than l needed to fully specify all irreducible representations, and in addition, it labels the different functions within each irreducible representation.

Let us demonstrate this construction in two and three dimensions. In two dimensions σ is not needed since all the irreducible representations are one dimensional and are spanned by $Y_l(\hat{u})=e^{il\phi}$ with ϕ being the angle between $\hat{u}$ and the the vector $\hat{e}_1 \equiv (1,0)$. Any rotation of the coordinates in an angle ϕ_0 results in a multiplicative factor $e^{i\phi_0}$. It is clear that $Y_l(\hat{u})$ is a polynomial in $\hat{u}$ since $Y_l(\hat{u})=[\hat{u}\cdot\hat{p}]^l$, where $\hat{p}\equiv(1,i)$. In three dimensions $\sigma=m$, where m takes on $2l+1$ values $m=-l,-l+1,...,l$. Here $Y_{l,m} \propto e^{im\phi}P_l^m(\cos\theta)$, where ϕ and θ are the usual spherical coordinates, and P_l^m is the associated Legendre polynomial of degree $l-m$. Obviously we again have a polynomial in $\hat{u}$ of degree l.

We now wish to calculate the action of the Laplacian operator with respect to u on the $Y_{l,\sigma}(\hat{u})$. We prove the following identity:

$$u^2\partial^\alpha\partial_\alpha Y_{l,\sigma}(\hat{u}) = -l(l+d-2)Y_{l,\sigma}(\hat{u}). \tag{A2}$$

One can easily check that for $d=3$, (A2) gives the factor $l(l+1)$, well known from the theory of angular momentum in quantum mechanics. To prove this identity for any d, note that

$$|u|^{2-l}\partial^2|u|^l Y_{l,\sigma}(\hat{u}) = 0. \tag{A3}$$

This follows from the fact that the Laplacian is an isotropic operator, and therefore is diagonal in the $Y_{l,\sigma}$. The same is true for the operator $|u|^{2-l}\partial^2|u|^l$. But this operator results in a polynomial in $\hat{u}$ of degree $l-2$, which is spanned by $Y_{l',\sigma'}$ such that $l'\leqslant l-2$. Therefore the RHS of (A3) must vanish. Accordingly we write

$$\partial^2|u^l|Y_{l,\sigma}(\hat{u}) + 2\partial^\alpha|u^l|\partial^\alpha Y_{l,\sigma} + |u^l|\partial^2 Y_{l,\sigma}(\hat{u}) = 0. \tag{A4}$$

The second term vanishes since it contains a radial derivative $u^\alpha\partial_\alpha$ operating on $Y_{l,\sigma}(\hat{u})$, which depends on $\hat{u}$ only. The first and third terms, upon elementary manipulations, lead to (A2).

Having the $Y_{l,\sigma}(\hat{u})$ we can now construct the irreducible representations in the space of constant tensors. The method is based on acting on the $Y_{l,\sigma}(\hat{u})$ with the isotropic operators u^α, ∂^α, and $\delta^{\alpha\beta}$. Due to the isotropy of the above operators, the behavior of the resulting expressions under rotations is similar to the behavior of the scalar function we started with. For example, the tensor fields $\delta^{\alpha\beta}Y_{l,\sigma}(\hat{u})$, $\partial^\alpha\partial^\beta Y_{l,\sigma}(\hat{u})$ transform under rotations according to the (l,σ) sector of SO(d).

Next, we wish to find the basis for the irreducible representations of the space of constant and fully symmetric tensors with n indices. We form the basis

$$B^{\alpha_1,\ldots,\alpha_n}_{l,\sigma,n} \equiv \partial^{\alpha_1}\cdots\partial^{\alpha_n}u^n Y_{l,\sigma}(\hat{u}), \quad l\leqslant n. \tag{A5}$$

Note that when l and n are even (as is the case invariably in this paper), $B^{\alpha_1,\ldots,\alpha_n}_{l,\sigma,n}$ no longer depends on $\hat{u}$, and is indeed fully symmetric by construction. Simple arguments can also prove that this basis is indeed complete, and spans all fully symmetric tensors with n indices. Other examples of this procedure for the other spaces are presented directly in the text.

Finally let us introduce two identities involving the $B_{n,l,\sigma}$ that are used over and over through the paper. The first one is

$$\delta_{\alpha_1\alpha_2}B^{\alpha_1,\ldots,\alpha_n}_{l,\sigma,n} = z_{n,l}B^{\alpha_3,\ldots,\alpha_n}_{l,\sigma,n-2}, \tag{A6}$$

$$z_{n,l} = [n(n+d-2)-l(l+d-2)]. \tag{A7}$$

It is straightforward to derive this identity using (A2). The second identity is

$$\sum_{i\neq j}\delta^{\alpha_i\alpha_j}B^{\{\alpha_m\},m\neq i,j}_{l,\sigma,n-2} = B^{\alpha_1,\ldots,\alpha_n}_{l,\sigma,n}, \quad l\leqslant n-2\ . \tag{A8}$$

This identity is proven by writing u^n in (A5) as u^2u^{n-2}, and operating with the derivative on u^2. The term obtained as $u^2\partial^{\alpha_1}\cdots\partial^{\alpha_n}u^{n-2}Y_{l,\sigma}(\hat{u})$ vanishes because we have n derivatives on a polynomial of degree $n-2$. It is worthwhile no-

ticing that these identities connect tensors from two different spaces: The space of tensors with n indices and the space of tensors with $n-2$ indices. Nevertheless, in both spaces, the tensors belong to the same (l,σ) sector of the SO(d) group. This is due to the isotropy of the contraction with $\delta^{\alpha_1\alpha_2}$ in the first identity, and the contraction with $\delta^{\alpha_i\alpha_j}$ in the second identity.

APPENDIX B: PROOF OF EQ. (3.32)

In case a of region II where $p_1>p_2>k_s$ the analytic expression for $\mathbf{A}_2$ can be simplified to

$$A^{\alpha_i\alpha_j}_{2\beta_i\beta_j}=\frac{1}{2}\hat{p}_1^{\alpha_i}\hat{p}_1^{\alpha_j}\int\frac{d\hat{p}_2}{\Omega_d}p^{\beta_i\beta_j}(\hat{p}_2). \tag{B1}$$

Using the identities

$$\int d\hat{p}=\Omega(d),\quad \Omega_d\equiv\Omega(d)\frac{d-1}{d}, \tag{B2}$$

$$\int d\hat{p}\hat{p}^{\alpha}\hat{p}^{\beta}=\delta_{\alpha\beta}\Omega(d), \tag{B3}$$

we compute

$$A^{\alpha_i\alpha_j}_{2\beta_i\beta_j}=\frac{1}{2}\hat{p}_1^{\alpha_i}\hat{p}_1^{\alpha_j}\delta_{\beta_i\beta_j}. \tag{B4}$$

Substituting this form into the double rung ladder diagram results, after contracting all the indices of $\mathbf{A}_1$ and $\mathbf{A}_2$, in a form identical to Eq. (3.15) for $\mathbf{A}$ in the one rung ladder diagram. This leads directly to the final equation, Eq. (3.31).

APPENDIX C: DOUBLE RESUMMATION

1. Calculation of $D_{m,s}$

In this Appendix we discuss the calculation of the coefficients $D_{m,s}$ in Eq. (3.34), and the actual resummation of that equation.

First, we need to introduce rules to evaluate the rungs in the general ladder diagram that appear in the expansion. The rule is actually quite simple: every rung contributes a term proportional to $gA_n^{(l)}$ if the $\mathbf{p}$ vector associated with this rung is the largest among all the $\mathbf{p}$ vectors associated with rungs appearing to the right of it. Otherwise the contribution is proportional to $\tilde{g}$. The weight of the contribution is obtained as a factor $c\leqslant 1$ which reflects the proportional fraction of the volume of $(\mathbf{p}_1,\mathbf{p}_2,\ldots)$ space in which the associated ordering of the $\mathbf{p}$ vectors is valid. For example, if the rung with the largest $\mathbf{p}$ vector is in the extreme right, then all the other rungs contribute terms proportional to $\tilde{g}$. Thus a diagram with s rungs ordered in this manner contributes with a weight $C=1/s$. Therefore

$$D_{1,s}=\frac{1}{s}. \tag{C1}$$

The recurrence relation for $D_{m,s}$ with $m>1$ is derived by inserting an additional rung that is associated with the largest $\mathbf{p}$ vector in any one of the $(s+1)$ possible positions available in a diagram with s rungs. After some combinatorial calculations of the weights one finds

$$D_{m+1,s}=\frac{1}{s}\sum_{q=m}^{s-1}D_{q,m}. \tag{C2}$$

Together with (C1) this gives

$$D_{2,s}=\frac{1}{s}\sum_{q=1}^{s-1}\frac{1}{q}, \tag{C3}$$

$$D_{3,s}=\frac{1}{s}\sum_{q_2=2}^{s-1}\frac{1}{q_2}\sum_{q_1=1}^{q_2-1}\frac{1}{q_1}, \tag{C4}$$

$$D_{4,s}=\frac{1}{s}\sum_{q_3=3}^{s-1}\frac{1}{q_3}\sum_{q_2=2}^{q_3-1}\frac{1}{q_2}\sum_{q_1=1}^{q_2-1}\frac{1}{q_1},\ \text{etc.} \tag{C5}$$

The general structure of $D_{m,s}$ now becomes obvious.

2. Higher order terms in A

Consider the equation (3.37)

$$K_2(g,A)=A^2\sum_{s=2}^{\infty}\frac{\tilde{g}^s}{s}\sum_{q=1}^{s-1}\frac{1}{q}. \tag{C6}$$

Observing that

$$\sum_{q=1}^{s-1}\frac{1}{q}=\frac{1}{2}\left[\sum_{q=1}^{s-1}\left(\frac{1}{s-q}+\frac{1}{q}\right)\right], \tag{C7}$$

and

$$\frac{1}{s}\left(\frac{1}{q}+\frac{1}{s-q}\right)=\frac{1}{q(s-q)}, \tag{C8}$$

we end up with

$$K_2(\tilde{g},A)=\frac{1}{2}A^2\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\frac{\tilde{g}^{q_1+q_2}}{q_1q_2}, \tag{C9}$$

where we relabeled $q\to q_1$ and $s-q\to q_2$ and changed correspondingly the limit of summation over q_2. Thus

$$K_2(\tilde{g},A)=\frac{1}{2}A^2\left[\sum_{q=1}^{\infty}\frac{1}{q}\right]^2=\frac{1}{2}[-A\ln(1-g)]^2. \tag{C10}$$

The terms proportional to A^3 give

$$K_3(\tilde{g},A)=A^3\sum_{s=3}^{\infty}\frac{\tilde{g}^n}{n}\sum_{q_2=2}^{s-1}\frac{1}{q_2}\sum_{q_1=1}^{q_2-1}\frac{1}{q_1}. \tag{C11}$$

We can rearrange the sums by summing over $q_3=n-q_1-q_2$ instead of n. Using relationships similar to (C7) and (C8) we find

$$K_3=\frac{1}{6}A^2\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\sum_{q_3=1}^{\infty}\frac{\tilde{g}^{q_1+q_2+q_3}}{q_1q_2q_3}. \tag{C12}$$

Obviously this leads to

$$K_3(\tilde{g},A)=\frac{1}{6}A^3\left[\sum_{q=1}^{\infty}\frac{\tilde{g}^q}{q}\right]^3=\frac{1}{3!}[-A\ln(1-\tilde{g})]^3. \tag{C13}$$

The general structure is now clear, leading to Eq. (3.38).

APPENDIX D: DERIVATION OF THE FUSION RULES

In this Appendix we derive the fusion rules (4.25). Consider a fully unfused structure function with n coordinates, such that p of them are separated from each other by a typical distance r, whereas $n-p$ coordinates are separated from them and from each other by a typical distance R, and $R \gg r$. We want to compute the asymptotic properties of S_n and show that to leading order in r/R we find Eq. (4.25). For homogeneous ensembles we can shift the origin to the center of mass of the p coordinates. In this case we have $r_j \ll r_i$ for every $j \leq p$ and $i>p$. Our aim is to separate the dependence on the small distances from the dependence on the large distances. We will see that some of the terms in S_n lend themselves naturally to such a separation, and some call for more work. We start from Eq. (4.22), and compute to first order in r/R:

$$\begin{aligned}
&f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)\\
&\quad\equiv(r_i-r_j)^{\alpha_i}(r_i-r_j)^{\alpha_j}\ln|\mathbf{r}_i-\mathbf{r}_j|\\
&\qquad+(\bar{r}_i-\bar{r}_j)^{\alpha_i}(\bar{r}_i-\bar{r}_j)^{\alpha_j}\ln|\bar{\mathbf{r}}_i-\bar{\mathbf{r}}_j|-(r_i-\bar{r}_j)^{\alpha_i}\\
&\qquad\times(r_i-\bar{r}_j)^{\alpha_j}\ln|\mathbf{r}_i-\bar{\mathbf{r}}_j|-(\bar{r}_i-r_j)^{\alpha_i}(\bar{r}_i-r_j)^{\alpha_j}\ln|\bar{\mathbf{r}}_i-\mathbf{r}_j|\\
&\quad=\left[-2r_i^{\alpha_i}\delta_\beta^{\alpha_j}\ln r_i-r_i^{\alpha_i}r_i^{\alpha_j}\frac{(r_i)_\beta}{r_i^2}+2\bar{r}_i^{\alpha_i}\delta_\beta^{\alpha_j}\ln\bar{r}_i\right.\\
&\qquad\left.+\bar{r}_i^{\alpha_i}\bar{r}_i^{\alpha_j}\frac{(\bar{r}_i)_\beta}{\bar{r}_i^2}\right]\Delta_j^\beta\equiv g^{\alpha_i\alpha_j}\beta(\mathbf{r}_i,\bar{\mathbf{r}}_i)\Delta_j^\beta,
\end{aligned} \tag{D1}$$

and so, if $r_j,\bar{r}_j \ll r_i,\bar{r}_i$ for $j=1,\ldots,p$, $i=p+1,\ldots,n$ then the first order in ϵ of $\mathcal{S}_n$ will contain three types of terms:

$$\begin{aligned}
I_1&=\sum_{1\leq i\neq j\leq p}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)[aB_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n}\\
&\quad+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}]\\
&=\sum_{1\leq i\neq j\leq p}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)[aB_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n}\\
&\quad+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}]\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n},
\end{aligned}$$

$$\begin{aligned}
I_2&=\sum_{1\leq j\leq p<i\leq n}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)\\
&\quad\times[aB_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}]\\
&=\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p}\sum_{1\leq j\leq p,p<i\leq n}\overbrace{\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i}g^{\alpha_i\beta}\alpha_j(\mathbf{r}_i,\bar{\mathbf{r}}_i)\\
&\quad\times[aB_{n,l,\sigma}^{\alpha_1,\ldots,\beta,\ldots,\alpha_n}+b_{n,l}\delta^{\alpha_i\beta}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}],
\end{aligned}$$

$$\begin{aligned}
I_3&=\sum_{p\leq i,j\leq n}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)[aB_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n}\\
&\quad+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}]\\
&=\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p}\sum_{p\leq i,j\leq n}\overbrace{\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)\\
&\quad\times[aB_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}].
\end{aligned}$$

We note that of these three terms only I_2 has a nontrivial mixing of small and large coordinates, and indeed it is the only term in which the expansion (D1) was employed. Collecting terms we find

$$\begin{aligned}
S_n(\mathbf{r}_1,\bar{\mathbf{r}}_1;\ldots;\mathbf{r}_n,\bar{\mathbf{r}}_n)&=\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p}\tilde{B}_p^{\alpha_1,\ldots,\alpha_p}\\
&\quad+\epsilon\sum_{i\neq j}\overbrace{\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p}}^{\text{no } i,j}f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)\\
&\quad\times[a\tilde{B}_p^{\alpha_1,\ldots,\alpha_p}+b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{\tilde{B}_{p-2}^{\alpha_1,\ldots,\alpha_p}}^{\text{no } i,j}]\\
&\quad+\epsilon\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p}\tilde{C}_p^{\alpha_1,\ldots,\alpha_p}+O(\epsilon^2),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{B}_p^{\alpha_1},\ldots,\alpha_p&=\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}B_{n,\ldots,\sigma}^{\alpha_1\ldots\alpha_n}\\
&=\sum_{j=\max\{0,p+l-n\}}^{p}\sum_{\sigma'}c_{j,\sigma'}B_{p,j,\sigma}^{\alpha_1,\ldots,\alpha_p},
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_{p-2}^{\alpha_3\ldots\alpha_p}&=\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}B_{n-2,l,\sigma}^{\alpha_3,\ldots,\alpha_n}\\
&=\sum_{j=\max\{0,p+l-n\}}^{p-2}\sum_{\sigma'}d_{j,\sigma'}B_{p-2,j,\sigma}^{\alpha_1,\ldots,\alpha_p},
\end{aligned}$$

$$\tilde{C}_p^{\alpha_1,\ldots,\alpha_p} = \sum_{p\leqslant i,j\leqslant n} \overbrace{\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j} f^{\alpha_i\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i,\mathbf{r}_j,\bar{\mathbf{r}}_j)$$
$$\times[aB_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n} + b_{n,l}\delta^{\alpha_i\alpha_j}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}]$$
$$+ \sum_{1\leqslant j\leqslant p, p<i\leqslant n} \overbrace{\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}}^{\text{no } i,j} g^{\alpha_i\beta}{}_{\alpha_j}(\mathbf{r}_i,\bar{\mathbf{r}}_i)$$
$$\times[aB_{n,l,\sigma}^{\alpha_1,\ldots,\beta,\ldots,\alpha_n} + b_{n,l}\delta^{\alpha_i\beta}\overbrace{B_{n-2,l,\sigma}^{\alpha_1,\ldots,\alpha_n}}^{\text{no } i,j}]$$
$$= \sum_{j=\max\{0,p+l-n\}}^{p} \sum_{\sigma'} e_{j,\sigma'} B_{p,j,\sigma'}^{\alpha_1,\ldots,\alpha_p}.$$

In these expressions we use the fact $\tilde{\mathbf{B}}_p$ is a fully symmetric tensor and therefore can be again expanded in terms of the basis functions $\mathbf{B}_{p,j,\sigma'}$ with coefficients that depend on the large separations. The sums on the right-hand sides run between $j=\max\{0,p+l-n\}$ and p because not all the basis functions can appear when $p+l-n>0$. This can be checked by contracting the $\tilde{\mathbf{B}}_p$ with $(n-l+2)/2$ δ function. This contraction vanishes since it contains a factor $z_{l,l}$. On the other hand, the contraction results in a tensor with $p+l-n-2$ indices, and therefore all corresponding coefficients of $j\leqslant p+l-n-2$ must vanish. To proceed we establish the following identity:

$$\frac{z_{p,j}}{z_{n,l}} c_{j,\sigma'} = d_{j,\sigma'}. \tag{D2}$$

The identity is proven by the following calculations:

$$\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n} B_{n,l,\sigma}^{\alpha_1,\ldots,\alpha_n}$$
$$= \partial^{\alpha_1}\cdots\partial^{\alpha_p} u^p u^{-p}[\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}]\partial_{\alpha_{p+1}}\cdots\partial_{\alpha_n} u^n Y_{l,\sigma}(\hat{u})$$
$$= \partial^{\alpha_1}\cdots\partial^{\alpha_p} u^p \sum_{j,\sigma'} c_{j,\sigma'} Y_{j,\sigma'}(\hat{u})$$

$$\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n} B_{n-2,l,\sigma}^{\alpha_3,\ldots,\alpha_n}$$
$$= \partial^{\alpha_3}\cdots\partial^{\alpha_p} u^{p-2} u^{-p+2}[\Delta_{p+1,}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}]$$
$$\times \partial_{\alpha_{p+1}}\cdots\partial_{\alpha_n} u^{n-2} Y_{l,\sigma}(\hat{u})$$
$$= \partial^{\alpha_3}\cdots\partial^{\alpha_p} u^p \sum_{j,\sigma'} d_{j,\sigma'} Y_{j,\sigma'}(\hat{u}).$$

Denote now

$$f(\hat{u}) \equiv u^{-p}[\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}]\partial_{p+1}\cdots\partial_n u^n Y_{l,\sigma}(\hat{u})$$
$$= \sum_{j,\sigma'} c_{j,\sigma'} Y_{j,\sigma'}(\hat{u}),$$

$$g(\hat{u}) \equiv u^{-p+2}[\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}]\partial_{p+1}\cdots\partial_n u^{n-2} Y_{l,\sigma}(\hat{u})$$
$$= \sum_{j,\sigma'} d_{j,\sigma'} Y_{j,\sigma'}(\hat{u}).$$

To obtain (D2), operate with $u^2\partial^2$ on $f(\hat{u})$. On the one hand, we get

$$u^2\partial^2 f(\hat{u}) = \sum_{j,\sigma'} -j(j+d-2)c_{j,\sigma'} Y_{j,\sigma'}(\hat{u})$$

but on the other hand, we have

$$u^2\partial^2 f(\hat{u}) = u^2\partial^2[u^{-p}[\Delta_{p+1}^{\alpha_{p+1}},\ldots,\Delta_n^{\alpha_n}]\partial_{p+1}\cdots\partial_n u^n Y_{l,\sigma}(\hat{u})]$$
$$= -p(-p+d-2)f(\hat{u}) - 2p^2 f(\hat{u}) + z_{n,l} g(\hat{u})$$
$$= -p(p+d-2)f(\hat{u}) + z_{n,l} g(\hat{u}).$$

Equating the two expressions, and projecting over the (j,σ') sector, we obtain

$$-j(j+d-2)c_{j,\sigma'} = -p(p+d-2)c_{j,\sigma'} + z_{n,l} d_{j,\sigma'}$$

$$d_{j,\sigma'} = \frac{[p(p+d-2)-j(j+2-2)]}{z_{n,l}} c_{j,\sigma'} = \frac{z_{p,j}}{z_{n,l}} c_{j,\sigma'}.$$

Recalling Eq. (4.21), $b_{p,j} = b_{n,l} z_{p,j}/z_{n,l}$ and we may write to leading order in r/R:

$$S_n(\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots;\mathbf{r}_n\overline{\mathbf{r}}_n)$$
$$= \sum_{j=\max\{0,p+l-n\}}^{p} \sum_{\sigma'} C_{j,\sigma'}\Bigg[\Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p} B_{p,j,\sigma'}^{\alpha_1,\ldots,\alpha_p}$$
$$+ \epsilon \sum_{i\neq j}^{\text{no } i,j} \Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p} f^{\alpha_i\alpha_j}(\mathbf{r}_i,\overline{\mathbf{r}}_i,\mathbf{r}_j,\overline{\mathbf{r}}_j)[aB_{p,l,\sigma'}^{\alpha_1,\ldots,\alpha_p}$$
$$+ b_{p,j}\delta^{\alpha_i\alpha_j} B_{p-2,l,\sigma'}^{\alpha_1,\ldots\alpha_p}]\Bigg] + \sum_{j=\max\{0,p+l-n\}}^{p} \sum_{\sigma'} \epsilon e_{j,\sigma'}$$
$$\times \Delta_1^{\alpha_1},\ldots,\Delta_p^{\alpha_p} B_{p,j,\sigma'}^{\alpha_1,\ldots,\alpha_p} + O(\epsilon^2)$$
$$= \sum_{j=\max\{0,p+l-n\}}^{p} \sum_{\sigma'} (c_{j,\sigma'} + \epsilon e_{j,\sigma'})$$
$$\times S_{p,j,\sigma'}(\mathbf{r}_1,\overline{\mathbf{r}}_1;\ldots;\mathbf{r}_p,\overline{\mathbf{r}}_p) + O(\epsilon^2).$$

From this follows Eq. (4.25).

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